Measure Theory with Ergodic Horizons
Lecture 15

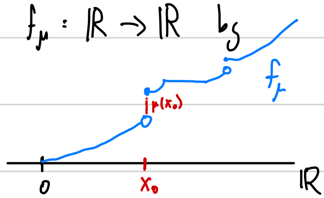
Locally finite Borel measures on $\mathbb{R}$.

To prove the measure isomorphism theorem, it is enough to analyze all Borel probability measures on $\mathbb{R}$, but we will do this more generally for all locally finite Borel measures.

Recall. Because $\mathbb{R}$ is locally compact and second ctbl, local finiteness of measures is equivalent to being finite on compact sets (hence bounded intervals have finite measure) and to being σ-finite by open sets. Thus, because $\mathbb{R}$ is a Polish metric space, these measures are strongly regular and tight.

Now let μ be a locally finite Borel measure on $\mathbb{R}$. Define $f_\mu : \mathbb{R} \to \mathbb{R}$ by

$$x \mapsto \begin{cases} \mu((0,x]) & \text{if } x \geq 0 \\ -\mu((x,0]) & \text{if } x < 0 \end{cases}.$$

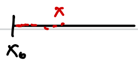


By definition, for each $a \leq b$ reals, we have $\mu((a,b]) = f_\mu(b) - f_\mu(a)$.
A function $f : \mathbb{R} \to \mathbb{R}$ with this property, i.e. $\mu((a,b]) = f(b) - f(a)$, is called the **distribution** of μ.

Prop. The distributions of μ are unique up to a constant, i.e. any two distributions f and g of μ satisfy $f = g + c$ for some $c \in \mathbb{R}$. Each distribution $f : \mathbb{R} \to \mathbb{R}$ of μ is

(i) increasing (non-decreasing)

(ii) right-continuous, i.e. $\lim_{x \searrow x_0^+} f(x) = f(x_0)$ for each $x_0 \in \mathbb{R}$.



(iii) continuous if μ is atomless.

Proof. For uniqueness up to a constant, note that for all $x \geq 0$,

$$f(x) = \mu((0,x]) + f(0) = g(x) - g(0) + f(0) = g(x) + c,$$

where $c := f(0) - g(0)$, for any two distributions f, g of μ. Same for $x < 0$.

(i) follows from the monotonicity of μ and (ii) follows from the decreasing monotonicity of finite measures: let $x_n \searrow x_0$, then $f(x_n) - f(x_0) = \mu((x_0, x_n]) < \infty$ and $\lim_{n\to\infty} \mu((x_0, x_n]) = \mu((x_0, x_0]) = 0$ by the decreasing monotone convergence.

(iii) is left as HW. $\square$

The converse is also true, which completes the characterization of all locally finite Borel measures on $\mathbb{R}$.

Theorem. Every increasing right-continuous function $f : \mathbb{R} \to \mathbb{R}$ is a distribution of a unique locally finite Borel measure μ on $\mathbb{R}$, given by, for all $a \leq b$,

$$\mu((a,b]) := f(b) - f(a).$$

Proof. The definition $\mu((a,b]) := f(b) - f(a)$ extends to a definition of a finitely additive measure on the algebra $\mathcal{A}$ of finite unions of intervals of the form $(a,b]$, where $a \in [-\infty, \infty)$ and $b \in (-\infty, \infty]$. This is done in the usual way by realizing that each set $A \in \mathcal{A}$ is a disjoint finite union of such half-open intervals $A = \bigsqcup_{i<n} I_i$, and that the definition

$$\mu(A) := \sum_{i<n} \mu(I_i)$$

doesn't depend on the representation $A = \bigsqcup_{i<n} I_i$, i.e. if we also have $A = \bigsqcup_{j<m} J_j$, then $\sum_{i<n} \mu(I_i) = \sum_{j<m} \mu(J_j)$. (We take a common refinement of $\{I_i\}$ and $\{J_j\}$.)

Having a finitely additive measure on $\mathcal{A}$, we then show that it is ctbly additive, and apply the Carathéodory extension theorem to get a measure μ on $\langle \mathcal{A} \rangle_\sigma$, which is equal to the Borel σ-algebra of $\mathbb{R}$ because each open interval $(a,b) = \bigcup_{n \in \mathbb{N}^+} (a, b - \frac{1}{n}]$. To show ctbl additivity, as before, we recall that finitely additive measures are automatically ctbly superadditive, so it remains to prove that μ is ctbly subadditive.

For this, let $(a,b]$ be a half-interval partitioned into half-intervals:

$$(a,b] = \bigsqcup_{n \in \mathbb{N}} (a_n, b_n].$$

We will prove that $\mu((a,b]) \leq \sum_{n \in \mathbb{N}} \mu((a_n, b_n])$ assuming that $(a,b]$ is bounded,

leaving the reduction of the unbounded case to the bounded one as an exercise.
We repeat the trick of making $(a,b]$ compact and the $(a_n, b_n]$ open as follows: using right-continuity of f, there are $a < a' \leq b$ and $b_n < b_n'$ such that $f(a) \approx_{\varepsilon/2} f(a')$ and $f(b_n) \approx_{\varepsilon/2^{n+2}} f(b_n')$. Now, $[a', b] \subseteq (a, b] = \bigsqcup_n (a_n, b_n] \subseteq \bigcup_n (a_n, b_n')$, so by the compactness of $[a', b]$, there is a finite $N \in \mathbb{N}$ such that

$$[a', b] \subseteq \bigcup_{n<N} (a_n, b_n'),$$

in particular,

$$(a', b] \subseteq \bigcup_{n<N} (a_n, b_n'],$$

so by finite subadditivity of μ, we have

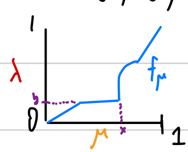
$$\mu((a,b]) \approx_{\varepsilon/2} \mu((a', b]) \leq \sum_{n<N} \mu((a_n, b_n']) \leq \sum_{n \in \mathbb{N}} \mu(a_n, b_n'] \approx_{\varepsilon/2} \sum_{n \in \mathbb{N}} \mu((a_n, b_n]).$$

This finishes the proof because ε is arbitrary. $\square$

We add one more property to the properties of distributions of a given μ.

Prop. For any atomless Borel probability measure μ on $[0,1]$, its distribution

$$f_\mu : [0,1] \to [0,1]$$
$$x \mapsto \mu((0,x])$$



is a measure isomorphism from $([0,1], \mu)$ to $([0,1], \lambda)$.
In particular, $(f_\mu)_* \mu = \lambda$.

Proof. For $(f_\mu)_* \mu = \lambda$, it is enough to show that these two measures agree on intervals $(0,y]$ because the measure of $(a,b]$ is the difference of the measures of $(0,b]$ and $(0,a]$. But $(f_\mu)_*\mu((0,y]) = \mu(f_\mu^{-1}(0,y])$. But f is surjective because it is continuous and takes values 0 and 1 (by intermediate value theorem), so $\exists x \in [0,1]$ such that $f(x)=y$ and using the increasingness of f_μ, there is the largest such $x \in [0,1]$. For this x, we have $f_\mu^{-1}(0,y] = (0,x]$, so $(f_\mu)_*\mu((0,y]) = \mu((0,x]) = f_\mu(x) = y = \lambda((0,y])$.

Let Z be the union of all maximal intervals $(a,b] \subseteq [0,1]$ such that $f(a)=f(b)$, i.e. $\mu((a,b])=0$. These intervals are disjoint and hence only ctbly many (because every interval contains a rational that others don't contain). Thus, Z is a ctbl union of μ-null set and hence $\mu(Z)=0$. It remains to verify that $f|_{[0,1]\setminus Z}$ is a bijection from $[0,1]\setminus Z$ to $[0,1]$. $\square$

Corollary. For any atomless Borel probability measure μ on $[0,1]$ there is a Borel isomorphism $f:[0,1)\to[0,1]$ such that $f_*\mu = \lambda$.

Proof. Let f_μ and Z be as in the previous proof and let $C \subseteq [0,1]$ be the standard $\frac{1}{3}$-Cantor set, in particular, $\lambda(C)=0$. Let $Y := f_\mu^{-1}(C)$, so $\mu(Y)=0$ because $\mu(Y) = (f_\mu)_*\mu(C) = \lambda(C) = 0$. This Y is closed because f_μ is continuous, and Z is Borel, so $Y \cup Z$ is Borel and unctbl. By a bit of Descriptive Set Theory, one can show that $(Y\cup Z, \mathcal{B}(Y\cup Z))$ is a standard Borel space,

hence the Borel isomorphism theorem gives a Borel isomorphism $g: Y \cup Z \to C$.
We finally define $f: [0,1] \to [0,1]$ by $f|_{[0,1] \setminus (Y \cup Z)} := f_\mu$ and $f|_{Y \cup Z} := g$. $\square$